

# Complete intersections in certain affine and projective monomial curves \*

Isabel Bermejo, Ignacio García-Marco

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## Abstract

Let  $k$  be an arbitrary field, the purpose of this work is to provide families of positive integers  $\mathcal{A} = \{d_1, \dots, d_n\}$  such that either the toric ideal  $I_{\mathcal{A}}$  of the affine monomial curve  $\mathcal{C} = \{(t^{d_1}, \dots, t^{d_n}) \mid t \in k\} \subset \mathbb{A}_k^n$  or the toric ideal  $I_{\mathcal{A}^*}$  of its projective closure  $\mathcal{C}^* \subset \mathbb{P}_k^n$  is a complete intersection. More precisely, we characterize the complete intersection property for  $I_{\mathcal{A}}$  and for  $I_{\mathcal{A}^*}$  when:

- (a)  $\mathcal{A}$  is a generalized arithmetic sequence,
- (b)  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and  $d_n \in \mathbb{Z}^+$ ,
- (c)  $\mathcal{A}$  consists of certain terms of the  $(p, q)$ -Fibonacci sequence, and
- (d)  $\mathcal{A}$  consists of certain terms of the  $(p, q)$ -Lucas sequence.

The results in this paper arise as consequences of those in [3, 5] and some new results regarding the toric ideal of the curve.

## 1 Introduction

Let  $k$  be an arbitrary field and  $k[\mathbf{x}] = k[x_1, \dots, x_n]$  and  $k[\mathbf{t}] = k[t_1, \dots, t_m]$  two polynomial rings over  $k$ . A *binomial*  $f$  in  $k[\mathbf{x}]$  is a difference of two monomials. Let  $\mathcal{B} = \{b_1, \dots, b_n\}$  be a set of nonzero vectors in  $\mathbb{N}^m$ ; each vector  $b_i = (b_{i1}, \dots, b_{im})$  corresponds to a monomial  $\mathbf{t}^{b_i} = t_1^{b_{i1}} \dots t_m^{b_{im}}$  in  $k[\mathbf{t}]$ .

The *toric ideal* determined by  $\mathcal{B}$  is the kernel of the homomorphism of  $k$ -algebras

$$\varphi: k[\mathbf{x}] \rightarrow k[\mathbf{t}]; \quad x_i \mapsto \mathbf{t}^{b_i}$$

and is denoted by  $I_{\mathcal{B}}$ . By [21, Corollary 4.3],  $I_{\mathcal{B}}$  is an  $\mathcal{B}$ -homogeneous binomial ideal, i.e., if one sets the  $\mathcal{B}$ -degree of a monomial  $\mathbf{x}^{\alpha} \in k[\mathbf{x}]$  as  $\deg_{\mathcal{B}}(\mathbf{x}^{\alpha}) := \alpha_1 b_1 + \dots + \alpha_n b_n \in \mathbb{N}^m$ , and says that a polynomial  $f \in k[\mathbf{x}]$  is  $\mathcal{B}$ -homogeneous if its monomials have the same  $\mathcal{B}$ -degree, then  $I_{\mathcal{B}}$  is generated by  $\mathcal{B}$ -homogeneous binomials. According to [21, Lemma 4.2], the height of  $I_{\mathcal{B}}$  is  $\text{ht}(I_{\mathcal{B}}) = n - \text{rk}(\mathbb{Z}\mathcal{B})$ , where  $\text{rk}(\mathbb{Z}\mathcal{B})$  denotes the rank of the subgroup

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of  $\mathbb{Z}^m$  generated by  $\mathcal{B}$ . The ideal  $I_{\mathcal{B}}$  is a *complete intersection* if  $\mu(I_{\mathcal{B}}) = \text{ht}(I_{\mathcal{B}})$ , where  $\mu(I_{\mathcal{B}})$  denotes the minimal number of generators of  $I_{\mathcal{B}}$ . Equivalently,  $I_{\mathcal{B}}$  is a complete intersection if there exists a set of  $s = n - \text{rk}(\mathbb{Z}\mathcal{B})$   $\mathcal{B}$ -homogeneous binomials  $g_1, \dots, g_s$  such that  $I_{\mathcal{A}} = (g_1, \dots, g_s)$ .

Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be a subset of the positive integers and consider the *affine monomial curve*  $\mathcal{C}$  given parametrically by  $x_1 = t^{d_1}, \dots, x_n = t^{d_n}$ , i.e.,

$$\mathcal{C} = \{(t^{d_1}, \dots, t^{d_n}) \in \mathbb{A}_k^n \mid t \in k\}.$$

By [20, Corollary 7.1.12], if  $k$  is an infinite field the ideal  $I(\mathcal{C})$  of polynomials vanishing on  $\mathcal{C}$  is equal to  $I_{\mathcal{A}}$ , the toric ideal determined by  $\mathcal{A}$ .  $I_{\mathcal{A}}$  is called the *toric ideal of  $\mathcal{C}$* .

Set  $d := \max(\mathcal{A})$  and consider  $\mathcal{A}^* := \{a_1, \dots, a_{n-1}, a_n, a_{n+1}\} \subset \mathbb{N}^2$ , where  $a_i := (d_i, d - d_i)$  for all  $i \in \{1, \dots, n\}$  and  $a_{n+1} := (0, d)$ , and the *projective monomial curve*  $\mathcal{C}^*$  obtained as the projective closure of  $\mathcal{C}$ , which is given parametrically by  $x_1 = t^{d_1}u^{d-d_1}, \dots, x_n = t^{d_n}u^{d-d_n}, x_{n+1} = u^d$ , i.e.,

$$\mathcal{C}^* = \{(t^{d_1}u^{d-d_1} : \dots : t^{d_n}u^{d-d_n} : u^d) \in \mathbb{P}_k^n \mid (t : u) \in \mathbb{P}_k^1\}.$$

Again by [20, Corollary 7.1.12], if  $k$  is an infinite field, then  $I(\mathcal{C}^*) = I_{\mathcal{A}^*}$ . The toric ideal  $I_{\mathcal{A}^*}$  is homogeneous, indeed  $I_{\mathcal{A}^*}$  is the homogenization of  $I_{\mathcal{A}}$  with respect to the variable  $x_{n+1}$ .  $I_{\mathcal{A}^*}$  is called the *toric ideal of  $\mathcal{C}^*$* .

Both  $I_{\mathcal{A}}$  and  $I_{\mathcal{A}^*}$  have height  $n - 1$ ; thus  $I_{\mathcal{A}}$  (resp.  $I_{\mathcal{A}^*}$ ) is a *complete intersection* if there exists a system of  $\mathcal{A}$ -homogeneous (resp. homogeneous) binomials  $g_1, \dots, g_{n-1}$  such that  $I_{\mathcal{A}} = (g_1, \dots, g_{n-1})$  (resp.  $I_{\mathcal{A}^*} = (g_1, \dots, g_{n-1})$ ). Clearly,  $I_{\mathcal{A}}$  is a complete intersection whenever  $I_{\mathcal{A}^*}$  is, the converse is not true in general.

The aim of this work is to provide families of positive integers  $\mathcal{A} = \{d_1, \dots, d_n\}$  such that either the toric ideal  $I_{\mathcal{A}}$  or  $I_{\mathcal{A}^*}$  is a complete intersection. The starting point are the papers by García-Sánchez and Rosales [12], Maloo and Sengupta [16] and Fel [11].

In the first one, the authors prove that if  $\mathcal{A}$  is a set of consecutive positive integers, then  $I_{\mathcal{A}}$  is a complete intersection if and only if  $n = 2$  or  $n = 3$  and  $d_1$  is even. In the second one, the authors obtain the same characterization when  $\mathcal{A}$  is an arithmetic sequence provided  $\gcd(\mathcal{A}) = 1$ . In Theorem 3.5 we generalize this result to *generalized arithmetic sequences*. We recall that  $\mathcal{A}$  is a generalized arithmetic sequence if there exists  $h \in \mathbb{Z}^+$  such that  $\{hd_1, d_2, \dots, d_n\}$  is an increasing arithmetic sequence.

Maloo and Sengupta also study the case in which  $\mathcal{A}$  is an almost-arithmetic sequence, i.e.,  $\mathcal{A} \setminus \{d_n\}$  is an arithmetic sequence and  $d_n \in \mathbb{Z}^+$ , and prove that  $n \leq 4$  provided  $I_{\mathcal{A}}$  is a complete intersection. In Theorem 3.9 we go further and characterize when  $I_{\mathcal{A}}$  is a complete intersection whenever  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and  $d_n \in \mathbb{Z}^+$ , where  $n \geq 4$ .

In the third one, the author provides certain conditions so that the semigroup generated by  $\mathcal{A} = \{d_1, d_2, d_3\}$  where  $\gcd\{d_1, d_2, d_3\} = 1$  and  $d_1, d_2, d_3$  are members of the Fibonacci or Lucas sequence is symmetric. Recall that for a set  $\mathcal{A} = \{d_1, \dots, d_n\}$  with  $\gcd(\mathcal{A}) = 1$ ,

setting  $\mathcal{S} := \sum_{i=1}^n \mathbb{N} d_i$ , then the complement  $\mathcal{S}$  in  $\mathbb{N}$  is finite and the largest integer not belonging to  $\mathcal{S}$  is called the *Frobenius number of  $\mathcal{S}$*  and denoted by  $g(\mathcal{S})$ . Moreover, the semigroup  $\mathcal{S}$  is *symmetric* if for every  $d \in \mathbb{Z}$ , either  $d \in \mathcal{S}$  or  $g(\mathcal{S}) - d \in \mathcal{S}$ . It is a classical result due to Herzog [13, Theorem 3.10] that whenever  $\mathcal{A} = \{d_1, d_2, d_3\} \subset \mathbb{Z}^+$  with  $\gcd\{d_1, d_2, d_3\} = 1$ , then  $\mathcal{S}$  is symmetric if and only if  $I_{\mathcal{A}}$  is a complete intersection. In this work we characterize the complete intersection property for  $I_{\mathcal{A}}$  when  $\mathcal{A}$  is a certain subset of either the  $(p, q)$ -Fibonacci sequence or the  $(p, q)$ -Lucas sequence, where  $p, q \in \mathbb{Z}^+$  are relatively prime. We recall that the  $(p, q)$ -Fibonacci sequence, denoted by  $\{F_n\}_{n \in \mathbb{N}}$ , is defined as follows

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = p F_{n+1} + q F_n \text{ for every } n \geq 0,$$

and the  $(p, q)$ -Lucas sequence, denoted by  $\{L_n\}_{n \in \mathbb{N}}$ , is defined as

$$L_0 = 2, L_1 = p \text{ and } L_{n+2} = p L_{n+1} + q L_n \text{ for every } n \geq 0.$$

These sequences are natural generalizations of the usual Fibonacci and Lucas sequences, now called  $(1, 1)$ -Fibonacci sequence and  $(1, 1)$ -Lucas sequence. In Theorems 4.5 and 5.1 we characterize when  $I_{\mathcal{A}}$  is a complete intersection, being  $\mathcal{A} = \{d_1, \dots, d_n\}$  with:

- (a)  $d_i = F_{e_i}$ , where  $\{e_1, \dots, e_n\}$  is a generalized arithmetic sequence, and
- (b)  $d_i = L_{e_i}$ , where  $\{e_1, \dots, e_n\}$  is an arithmetic sequence.

Moreover, we characterize algorithmically when the toric ideal of a projective monomial curve is a complete intersection with Algorithm CI-projective-monomial-curve (see Table 1). Using this algorithm we are able to characterize in Theorems 6.1, 6.3, 6.5 and 6.7, which are the projective versions of Theorems 3.5, 3.9, 4.5 and 5.1 respectively, when  $I_{\mathcal{A}^*}$  is a complete intersection when  $\mathcal{A}$  belongs to any of the four families already described.

Whenever  $I_{\mathcal{A}}$  or  $I_{\mathcal{A}^*}$  is a complete intersection, we also obtain a minimal set of generators of the toric ideal. Furthermore, when  $I_{\mathcal{A}}$  is a complete intersection, using the formula described in [8, Remark 11] and [2, Remark 4.5], the Frobenius number  $g(\mathcal{S})$  of the numerical semigroup  $\mathcal{S} := \sum_{i=1}^n \mathbb{N} (d_i/e)$ , where  $e := \gcd(\mathcal{A})$ , is also provided. Indeed, the formula asserts that if  $I_{\mathcal{A}} = (g_1, \dots, g_{n-1})$  where  $g_i$  is  $\mathcal{A}$ -homogeneous for all  $i \in \{1, \dots, n-1\}$ , then  $g(\mathcal{S}) = \left( \sum_{i=1}^{n-1} \deg_{\mathcal{A}}(g_i) - \sum_{i=1}^n d_i \right) / e$ .

The results obtained in this work arise as consequences of our papers [3, 5] and some new results concerning the toric ideal of the curves. In [3] we exploited the combinatorial-arithmetical structure of complete intersections given by the existence of a certain *binary tree labeled by  $\{d_1, \dots, d_n\}$*  stated in [2, Theorem 4.3] to obtain an algorithm that determines whether the toric ideal of an affine monomial curve is a complete intersection. In [5] we provided some new results concerning complete intersection toric ideals in general and apply them in order to obtain algorithms that characterize when either a simplicial toric ideal or a homogeneous simplicial toric ideal is a complete intersection. Since both  $I_{\mathcal{A}}$  and  $I_{\mathcal{A}^*}$  are simplicial toric ideals and, moreover,  $I_{\mathcal{A}^*}$  is homogeneous, the algorithms obtained in [5] apply to them. These algorithms have been implemented in ANSI C programming language and also in the distributed library `cisimplicial.lib` [4] of SINGULAR [7].

## 2 Complete intersection toric ideals associated to affine and projective monomial curves

This section is devoted to present some new results concerning the complete intersection property for toric ideals associated to either affine or projective monomial curves. The results of this section arise after applying some results of [5] to the context of affine and projective monomial curves. The main results of this section are namely Proposition 2.6 and Theorem 2.7. On one hand, Proposition 2.6 provides, under certain hypothesis, a necessary and sufficient condition for the toric ideal of an affine monomial curve to be a complete intersection. This result will be useful in Sections 3, 4 and 5. On the other hand, Theorem 2.7 is a particularization of [5, Corollary 5] for toric ideals associated to projective monomial curves. This result, together with Remark 2.8, yields Algorithm CI-projective-monomial-curve of Table 1, an algorithm for checking whether the toric ideal of a projective monomial curve is a complete intersection. This algorithm will be useful in Section 6.

In order to present the new results, we begin our explanation by briefly describing some results of [5]. For every set  $\mathcal{B} = \{b_1, \dots, b_n\}$  of nonzero vectors of  $\mathbb{N}^m$ , we have the following results.

**Lemma 2.1.** [5, Lemmas 2.1 and 2.2]

- If  $b_i \notin \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{Q} b_j$ , then  $I_{\mathcal{B}} = I_{\mathcal{B} \setminus \{b_i\}} \cdot k[\mathbf{x}]$ . Moreover,  $I_{\mathcal{B}}$  is a complete intersection  $\iff I_{\mathcal{B} \setminus \{b_i\}}$  so is.
- If  $b_i = \sum_{j \in \{1, \dots, n\}, j \neq i} \alpha_j b_j \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{N} b_j$ , then  $I_{\mathcal{B}} = I_{\mathcal{B} \setminus \{b_i\}} \cdot k[\mathbf{x}] + (x_i - \prod_{j \in \{1, \dots, n\}, j \neq i} x_j^{\alpha_j})$ . Moreover,  $I_{\mathcal{B}}$  is a complete intersection  $\iff I_{\mathcal{B} \setminus \{b_i\}}$  so is.

Whenever  $b_i \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{Q} b_j$  we denote  $B_i := \min\{B \in \mathbb{Z}^+ \mid B b_i \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{Z} b_j\}$  and have the following result.

**Proposition 2.2.** [5, Proposition 2.3] Assume that  $b_i \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{Q} b_j$  for some  $i \in \{1, \dots, n\}$  and set  $\mathcal{B}' := \{b_1, \dots, B_i b_i, \dots, b_n\}$  and  $\rho : k[\mathbf{x}] \rightarrow k[\mathbf{x}]$  the morphism induced by  $\rho(x_i) = x_i^{B_i}$ ,  $\rho(x_j) = x_j$  for every  $j \neq i$ . Then,  $I_{\mathcal{B}} = \rho(I_{\mathcal{B}'}) \cdot k[\mathbf{x}]$ . Moreover,  $I_{\mathcal{B}}$  is a complete intersection  $\iff I_{\mathcal{B}'}$  is a complete intersection.

Applying Lemma 2.1 and Proposition 2.2 iteratively, we can associate to  $\mathcal{B}$  a unique subset  $\mathcal{B}_{red} \subset \mathbb{N}^m$  which can be either empty or satisfies that  $\mathcal{B}_{red} = \{b'_1, \dots, b'_r\}$ , where  $r \leq n$  and  $b'_i \in \sum_{j \in \{1, \dots, r\}, j \neq i} \mathbb{Z} b'_j \setminus \sum_{j \in \{1, \dots, r\}, j \neq i} \mathbb{N} b'_j$  for all  $i \in \{1, \dots, r\}$ . As a consequence of this construction we have the following result.

**Theorem 2.3.** [5, Theorem 2.5]  $I_{\mathcal{B}}$  is a complete intersection  $\iff$  either  $\mathcal{B}_{red} = \emptyset$  or  $I_{\mathcal{B}_{red}}$  is a complete intersection.

Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be a set of  $n \geq 2$  positive integers, in this setting we have that  $B_i = \min\{B \in \mathbb{Z}^+ \mid Bd_i \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{Z}d_j\} = \gcd(\mathcal{A} \setminus \{d_i\}) / \gcd(\mathcal{A})$  for all  $i \in \{1, \dots, n\}$ . For  $n = 3$  we have the following result, which is essentially a rewriting of a classical Herzog's result [13] (see also [23, Proposition 3]).

**Proposition 2.4.** *If  $n = 3$ ,  $I_{\mathcal{A}}$  is a complete intersection  $\iff \mathcal{A}_{red} = \emptyset$ .*

For  $n > 3$  the same characterization does not hold. Nevertheless, under certain hypothesis, we prove in Proposition 2.6 an analogous characterization for  $I_{\mathcal{A}}$  to be a complete intersection. To present this result we define

$$m_i := \min \left\{ b \in \mathbb{Z}^+ \mid bd_i \in \sum_{j \in \{1, \dots, n\}, j \neq i} \mathbb{N}d_j \right\} \text{ for every } i \in \{1, \dots, n\},$$

and say that a binomial  $f \in I_{\mathcal{A}}$  is *critical with respect to  $x_i$*  if  $f = x_i^{m_i} - \prod_{j \in \{1, \dots, n\}, j \neq i} x_j^{\alpha_j}$ . Critical binomials were introduced by Eliahou in [9] and later studied by Alcántar and Villarreal in [1]

**Lemma 2.5.** *Let  $f_1, \dots, f_t$  be critical binomials with respect to  $x_{i_1}, \dots, x_{i_t}$  respectively, where  $1 \leq i_1 < \dots < i_t \leq n$ . If  $m_i d_i \neq m_j d_j$  for every  $1 \leq i < j \leq t$ , then there exists a set of binomials  $\mathfrak{B}$  minimally generating  $I_{\mathcal{A}}$  such that  $f_1, \dots, f_t \in \mathfrak{B}$ .*

**Proof.** Let  $\{g_1, \dots, g_{n-1}\}$  be a set of  $\mathcal{A}$ -homogeneous binomials generating  $I_{\mathcal{A}}$ . Let  $D_1, \dots, D_{n-1}$  be the  $\mathcal{A}$ -degrees of  $g_1, \dots, g_{n-1}$  respectively and suppose that  $f_1 \in I_{\mathcal{A}}$  is a critical binomial with respect to  $x_1$ , thus  $f_1 = x_1^{m_1} - \prod_{j \in \{2, \dots, n\}} x_j^{\alpha_j}$  for some  $\alpha_2, \dots, \alpha_n \in \mathbb{N}$ . Hence  $f_1 = q_1 g_1 + \dots + q_{n-1} g_{n-1}$ , where  $q_k \in k[\mathbf{x}]$  is an  $\mathcal{A}$ -homogeneous polynomial of degree  $m_1 a_1 - D_k \geq 0$  when  $q_k \neq 0$  for all  $k \in \{1, \dots, n-1\}$ . In particular, there exists  $k \in \{1, \dots, n-1\}$  such that  $q_k \neq 0$  and the image of  $g_k$  under the evaluation morphism which sends  $x_j$  to 0 for all  $j \neq 1$  is equal to  $x_1^{D_k/d_1}$ , we assume that  $k = 1$ . Thus  $g_1 = x_1^{D_1/d_1} - \mathbf{x}^\beta$ , where  $\mathbf{x}^\beta$  is a monomial of  $\mathcal{A}$ -degree  $D_1$  which does not involve the variable  $x_1$ , and hence  $D_1 \in \mathbb{Z}^+ d_1 \cap \sum_{j \in \{2, \dots, n\}} \mathbb{N}d_j$ . By the definition of  $m_1$  we get the equality  $m_1 a_1 = D_1$ , which implies that  $q_1 \in k$  and  $\{f_1, g_2, \dots, g_{n-1}\}$  is a minimal set of generators of  $I_{\mathcal{A}}$ . Iterating this argument, we get the result.  $\square$

**Proposition 2.6.** *If  $n-1$  integers from  $m_1 d_1, \dots, m_n d_n$  are different, then  $I_{\mathcal{A}}$  is a complete intersection  $\iff \mathcal{A}_{red} = \emptyset$ .*

**Proof.**  $(\Leftarrow)$  Follows from Theorem 2.3.  $(\Rightarrow)$  Assume that  $m_1 d_1, \dots, m_{n-1} d_{n-1}$  are all different, by Lemma 2.5 we get that  $I_{\mathcal{A}} = (f_1, \dots, f_{n-1})$  where  $f_i$  is a critical binomial with respect to  $x_i$  for every  $1 \leq i \leq n-1$ . We claim that there exists  $j \in \{1, \dots, n-1\}$  such that  $x_j$  does not appear in  $f_k$  for all  $k \in \{1, \dots, n-1\} \setminus \{j\}$ . Suppose this claim is false, then we consider the simple directed graph with vertex set  $\{1, \dots, n-1\}$  and arc set  $\{(j, k) \mid 1 \leq j, k \leq n-1, j \neq k \text{ and } x_j \text{ appears in } f_k\}$ ; since the out-degree of every vertex is greater or equal to one, there is a cycle in the graph. Suppose that the cycle is  $(1, 2, \dots, k, 1)$  with  $k \leq n-1$ , this means that  $(f_1, \dots, f_k) \subset (x_1, \dots, x_k)$ , so

$I_{\mathcal{A}} \subsetneq H := (x_1, \dots, x_k, f_{k+1}, \dots, f_{n-1})$  but this is not possible because  $I_{\mathcal{A}}$  is prime and  $n - 1 = \text{ht}(I_{\mathcal{A}}) < \text{ht}(H) \leq n - 1$ .

Thus there exists  $i \in \{1, \dots, n - 1\}$  such that  $x_i$  only appears in  $f_i$ , suppose  $i = 1$ . Now we write  $\gamma_i := m_i e_i - \sum_{\substack{j \in \{1, \dots, n\} \\ j \neq i}} \alpha_{i,j} e_j \in \mathbb{Z}^n$  for all  $i \in \{1, \dots, n - 1\}$ . By [10, Proposition 2.3],  $\{\gamma_1, \dots, \gamma_{n-1}\}$  is a  $\mathbb{Z}$ -basis for the kernel of the homomorphism  $\tau : \mathbb{Z}^n \rightarrow \mathbb{Z}$  induced by  $\tau(e_j) = d_j$ . By definition,  $B_1 d_1 = \sum_{j \in \{2, \dots, n\}} \beta_j d_j$  for some  $\beta_j \in \mathbb{Z}$ , so take  $\delta := B_1 e_1 - \sum_{j \in \{2, \dots, n\}} \beta_j e_j \in \ker(\tau)$ . Consequently  $m_1$  divides  $B_1$  and by definition  $B_1$  divides  $m_1$ , so  $B_1 d_1 = m_1 d_1 \in \sum_{j \in \{2, \dots, n\}} \mathbb{N} d_j$ . Now we have that  $\mathcal{A}_{red} = (\mathcal{A} \setminus \{d_1\})_{red}$ , and by Lemma 2.1 and Proposition 2.2 it follows that  $I_{\mathcal{A} \setminus \{d_1\}}$  is a complete intersection minimally generated by  $\{f_2, \dots, f_{n-1}\}$ ; repeating the same argument we conclude that  $\mathcal{A}_{red} = \emptyset$ .  $\square$

Concerning the case of projective monomial curves, we denote  $d := \max(\mathcal{A})$  and  $\mathcal{A}^* = \{a_1, \dots, a_{n+1}\}$  where  $a_i = (d_i, d - d_i)$  for every  $i \in \{1, \dots, n\}$  and  $a_{n+1} = (0, d)$ . Since toric ideals associated to projective monomial curves is a subfamily of homogeneous simplicial toric ideals, the following result, which is a particular case of [5, Corollary 3.4], holds.

**Theorem 2.7.**  $I_{\mathcal{A}^*}$  is a complete intersection  $\iff \mathcal{A}_{red}^* = \emptyset$ .

Moreover, the computation of  $\mathcal{A}_{red}^*$  is simpler than in the general case if we take into account the following properties which are easy to prove.

**Remark 2.8.** (1)  $B_i = \gcd(\mathcal{A} \setminus \{d_i\}) / \gcd(\mathcal{A})$  for all  $i \in \{1, \dots, n\}$ .

(2)  $\forall i \in \{1, \dots, n\} : B_i a_i \in \sum_{\substack{j \in \{1, \dots, n+1\} \\ j \neq i}} \mathbb{N} a_j \iff B_i d_i = \sum_{\substack{j \in \{1, \dots, n\} \\ j \neq i}} \alpha_j d_j \in \sum_{\substack{j \in \{1, \dots, n\} \\ j \neq i}} \mathbb{N} d_j$ ,  
where  $\sum_{\substack{j \in \{1, \dots, n\} \\ j \neq i}} \alpha_j \leq B_i$ . Thus, checking whether  $B_i a_i \in \sum_{\substack{j \in \{1, \dots, n+1\} \\ j \neq i}} \mathbb{N} a_j$  is reduced to determining if an integer belongs to a subsemigroup of  $\mathbb{N}$  with an extra condition.

(3) If  $i = n + 1$  or  $d_i = d = \max(\mathcal{A})$ , then  $B_i a_i \notin \sum_{\substack{j \in \{1, \dots, n\} \\ j \neq i}} \mathbb{N} a_j$ .

In Table 1 we show Algorithm CI-projective-monomial-curve, an algorithm which receives as input a set  $\mathcal{A} = \{d_1, \dots, d_n\} \subset \mathbb{Z}^+$  and determines whether  $I_{\mathcal{A}^*}$  is a complete intersection. This algorithm is essentially obtained in [6, Theorem 3.6].

### 3 Complete intersections and generalized arithmetic sequences

In this section we deal with the cases in which either  $\mathcal{A}$  is a generalized arithmetic sequence, or  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and  $d_n$  is a positive integer. We denote by  $\mathcal{S}$  the semigroup spanned by  $\mathcal{A}$  and assume that  $\mathcal{S}$  is a **numerical semigroup**, i.e.,  $\gcd(\mathcal{A}) = 1$ , and that  $\mathcal{S}$  is **minimally generated by  $\mathcal{A}$** . Note that if  $\mathcal{A}$  is a generalized arithmetic sequence and  $\gcd(\mathcal{A}) = 1$ , one can easily check that  $\mathcal{A}$  is a minimal set of generators of  $\mathcal{S}$  if and only if  $n \leq a_1$ .

As we mentioned in the introduction, Maloo and Sengupta proved in [16] that whenever  $\mathcal{A} \setminus \{d_n\}$  is an arithmetic sequence and  $n \geq 5$ , then  $I_{\mathcal{A}}$  is not a complete intersection. To

**Algorithm CI-projective-monomial-curve**

*Entrada* :  $\mathcal{A} = \{d_1, \dots, d_n\} \subset \mathbb{Z}^+$

*Salida* : TRUE o FALSE

```

 $d := \max\{d_1, \dots, d_n\}$ 
repeat
   $\mathcal{B} := \mathcal{A}$ 
  for all  $d_i \in \mathcal{A} \setminus \{d\}$  do
     $B_i := \gcd(\mathcal{A} \setminus \{d_i\}) / \gcd(\mathcal{A})$ 
    if  $B_i d_i = \sum_{\substack{d_j \in \mathcal{A} \\ j \neq i}} \alpha_j d_j \in \sum_{\substack{d_j \in \mathcal{A} \\ j \neq i}} \mathbb{N} d_j$  and  $\sum \alpha_j \leq B_i$  then
       $\mathcal{A} := \mathcal{A} \setminus \{d_i\}$ 
    end if
  end for
until  $(\mathcal{A} = \{d\})$  OR  $(\mathcal{A} = \mathcal{B})$ 
if  $\mathcal{A} = \{d\}$  then
  return TRUE
end if
return FALSE

```

Table 1: Algorithm **CI-projective-monomial-curve**

prove this they used the description of a minimal set of generators of  $I_{\mathcal{A}}$  when  $\mathcal{A} \setminus \{d_n\}$  is an arithmetic sequence obtained by Patil and Singh [18] (see also [17] for a shorter proof of the same result). Here we present a generalization of Maloo and Sengupta's result which does not require to obtain a description of a minimal set of generators of  $I_{\mathcal{A}}$ . More precisely, we prove that whenever  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and  $n \geq 5$ , then  $I_{\mathcal{A}}$  is not a complete intersection. In order to prove this result, we first introduce two results.

**Lemma 3.1.** *Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be a subset of  $\mathbb{Z}^+$ . If there exist  $1 \leq i < j \leq n$  such that:*

- $m_i d_i \neq m_j d_j$  and
- $m_i d_i = \sum_{\substack{k \in \{1, \dots, n\} \\ k \neq i}} \alpha_k d_k$ ,  $m_j d_j = \sum_{\substack{k \in \{1, \dots, n\} \\ k \neq j}} \beta_k d_k$  with  $\alpha_k, \beta_k \in \mathbb{N}$ ,  $\alpha_j \neq 0$  and  $\beta_i \neq 0$ ,

*then  $I_{\mathcal{A}}$  is not a complete intersection.*

**Proof.** Assume that  $I_{\mathcal{A}}$  is a complete intersection and set  $f_i := x_i^{m_i} - \prod_{k \neq i} x_k^{\alpha_k}$  and  $f_j := x_j^{m_j} - \prod_{k \neq j} x_k^{\beta_k}$ . By Lemma 2.5 there exist some binomials  $g_3, \dots, g_{n-1} \in I_{\mathcal{A}}$  such that  $I_{\mathcal{A}} = (f_i, f_j, g_3, \dots, g_{n-1})$ . Therefore  $I_{\mathcal{A}} \subsetneq J := (x_i, x_j, g_3, \dots, g_{n-1})$ , but this is not possible because  $I_{\mathcal{A}}$  is a prime ideal and  $n - 1 = \text{ht}(I_{\mathcal{A}}) < \text{ht}(J) \leq n - 1$ .  $\square$

**Proposition 3.2.** *If  $n \geq 4$  and  $\mathcal{A}$  contains a generalized arithmetic sequence with 4 elements, then  $I_{\mathcal{A}}$  is not a complete intersection.*

**Proof.** Suppose that  $\{d_1, d_2, d_3, d_4\} \subset \mathcal{A}$  is a generalized arithmetic sequence, i.e., there exists  $h \in \mathbb{Z}^+$  such that  $\{hd_1, d_2, d_3, d_4\}$  is an arithmetic sequence. Since  $\mathcal{A}$  is a minimal set of generators of  $\mathcal{S}$ , we have that  $m_i > 1$  for all  $1 \leq i \leq n$ . Moreover, the equalities  $2d_2 = hd_1 + d_3$  and  $2d_3 = d_2 + d_4$  prove that  $m_2 = m_3 = 2$ , hence  $I_{\mathcal{A}}$  is not a complete intersection by Lemma 3.1.  $\square$

It is worth pointing out that from [15, Theorem 2.5] one can deduce a weaker version of Proposition 3.2 which states that if  $\mathcal{A}$  contains an arithmetic sequence with 5 elements, then  $I_{\mathcal{A}}$  is not a complete intersection.

From Proposition 3.2 one directly derives the following two corollaries.

**Corollary 3.3.** *If  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and  $n \geq 5$ , then  $I_{\mathcal{A}}$  is not a complete intersection.*

**Corollary 3.4.** *If  $\mathcal{A}$  is a generalized arithmetic sequence and  $n \geq 4$ , then  $I_{\mathcal{A}}$  is not a complete intersection.*

Now we can proceed with the characterizations:

**Theorem 3.5.** *Let  $\mathcal{A}$  be a generalized arithmetic sequence with  $n \geq 3$ . Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 3$  and  $d_1$  is even.*

**Proof.** By Corollary 3.4 it only remains to study when  $n = 3$ . Let  $h \in \mathbb{Z}^+$  be such that  $\{ha_1, a_2, a_3\}$  is an arithmetic sequence. Since  $\gcd(\mathcal{A}) = 1$ , denoting  $d := d_3 - d_2$  we have that  $d_2 = hd_1 + d$ ,  $d_3 = hd_1 + 2d$  and  $\gcd\{d_1, d\} = 1$ . We separate two cases, if  $d_1$  is even, then  $B_2d_2 = 2d_2 = hd_1 + d_3 \in \mathbb{N}\{d_1, d_3\}$  and  $\mathcal{A}_{red} = \emptyset$ , thus by Proposition 2.4,  $I_{\mathcal{A}}$  is a complete intersection. If  $d_1$  is odd, then

- $B_1d_1 = \gcd\{h, d\}d_1 < d_2 < d_3$ , thus  $B_1d_1 \notin \mathbb{N}\{d_2, d_3\}$ ,
- $B_2d_2 = d_2 \notin \mathbb{N}\{d_1, d_3\}$  and
- $B_3d_3 = d_3 \notin \mathbb{N}\{d_1, d_2\}$ .

So,  $\mathcal{A}_{red} = \{B_1d_1, d_2, d_3\}$  and again by Proposition 2.4 we conclude that  $I_{\mathcal{A}}$  is not a complete intersection.  $\square$

**Remark 3.6.** *Furthermore, whenever  $I_{\mathcal{A}}$  is a complete intersection, i.e., when  $\{d_1, d_2, d_3\}$  is a generalized arithmetic sequence and  $d_1$  is even, we get the following additional information (see Lemma 2.1 and Proposition 2.2):*

- $I_{\mathcal{A}} = \left(x_2^2 - x_1^h x_3, x_1^{d_3/2} - x_3^{d_1/2}\right)$ , with  $h \in \mathbb{Z}^+$  such that  $\{hd_1, d_2, d_3\}$  is an arithmetic sequence, i.e.,  $h = (2d_2 - d_3)/d_1$ .



- $g(\mathcal{S}) = d_1 d_3 / 2 - d_1 + d_2 - d_3$ .

The general formula for the Frobenius number of the semigroup  $\mathbb{N}\mathcal{A}$  when  $\mathcal{A}$  is a generalized arithmetic sequence can be found in [19, Theorem 3.3.4].

As a direct consequence of Theorem 3.5 we get the already cited results:

**Corollary 3.7.** [16, Theorem 3.5] *Let  $\mathcal{A}$  be an arithmetic sequence. Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 2$  or  $n = 3$  and  $d_1$  is even.*

**Corollary 3.8.** [12, Corollary 9] *Let  $\mathcal{A}$  be a set of consecutive integers. Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 2$  or  $n = 3$  and  $d_1$  is even.*

Concerning the case where  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence with  $n \geq 4$ , we have the following:

**Theorem 3.9.** *Let  $\mathcal{A} \setminus \{d_n\}$  be a generalized arithmetic sequence with  $n \geq 4$ . Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 4$  and one of the following holds:*

1.  $d_1 / \gcd\{d_1, d_2\}$  is even and  $\gcd\{d_1, d_2\}d_4 \in \mathbb{N}\{d_1, d_2, d_3\}$ , or
2.  $d_1, d_4$  are even and  $\mathcal{C}_{red} = \emptyset$  with  $\mathcal{C} = \{d_1, d_3, d_4\}$

**Proof.** By Corollary 3.3 it only remains to study when  $n = 4$ . By Proposition 2.2 we have that  $I_{\mathcal{A}}$  is a complete intersection if and only if  $I_{\mathcal{A}'}$  so is, where  $\mathcal{A}' := \{d_1, d_2, d_3, B_4 d_4\}$ . Moreover,  $I_{\mathcal{A}'} = I_{\mathcal{B}}$  where  $\mathcal{B} := \{d'_1, d'_2, d'_3, d'_4\}$  with  $d'_i = d_i / B_4$  for  $1 \leq i \leq 3$  and  $d'_4 = d_4$ .

We separate two cases, if  $d'_4 \in \mathbb{N}\{d'_1, d'_2, d'_3\}$ , then by Lemma 2.1,  $I_{\mathcal{A}}$  is a complete intersection if and only if  $I_{\mathcal{B} \setminus \{d'_4\}}$  so is. Since  $\mathcal{B} \setminus \{d'_4\} = \{d'_1, d'_2, d'_3\}$  is a generalized arithmetic sequence and  $\gcd\{d'_1, d'_2, d'_3\} = 1$ , by Theorem 3.5 we get that  $I_{\mathcal{A}}$  is a complete intersection if and only if  $d'_1$  is even. Assume now that  $d'_4 \notin \mathbb{N}\{d'_1, d'_2, d'_3\}$  and set  $m_i := \min\{b \in \mathbb{Z}^+ \mid bd'_i \in \mathbb{N}(\mathcal{B} \setminus \{d'_i\})\}$  for all  $1 \leq i \leq 4$ , then  $m_i \geq 2$  for every  $1 \leq i \leq 4$ , in particular  $m_2 = 2$ . Let us study the possible values of  $m_1$ , if  $m_1 d'_1 = m_2 d'_2$  we set  $d'_5 := \gcd\{d'_1, d'_2\} = 1$  and  $m'_3 := \min\{b \in \mathbb{Z}^+ \mid bd'_3 \in \mathbb{N}\{d'_4, d'_5\}\} = 1$  and by [3, Proposition 2.1] we have that  $I_{\mathcal{B}}$  is not a complete intersection because  $m_3 \neq m'_3$ . If  $m_1 d'_1 = m_3 d'_3$ , then necessarily  $d'_1$  is even; otherwise  $m_1 d'_1 = \text{lcm}\{d'_1, d'_3\} = d'_1 d'_3 > d'_1 d'_2$ , which contradicts the definition of  $m_1$ . Then we set  $d'_5 := \gcd\{d'_1, d'_3\} = 2$  and  $m'_i := \min\{b \in \mathbb{Z}^+ \mid bd'_i \in \sum_{j \in \{2,4,5\}, j \neq i} \mathbb{N}d'_j\}$  for  $i = 2, 4$ , and observe that  $m'_2 = 1$  or  $m'_4 = 1$ . Indeed, if  $d'_2$  is even then  $m'_2 = 1$ , if  $d'_4$  is even then  $m'_4 = 1$  and if both are odd, then  $m'_4 = 1$  if  $d'_2 < d'_4$ , or  $m'_2 = 1$  otherwise. Again by [3, Proposition 2.1] we have that  $I_{\mathcal{B}}$  is not a complete intersection. If  $m_1 d'_1 \notin \{m_2 d'_2, m_3 d'_3\}$  then  $m_1 d'_1, m_2 d'_2$  and  $m_3 d'_3$  are all different and by Proposition 2.6,  $I_{\mathcal{B}}$  is a complete intersection if and only if  $\mathcal{B}_{red} = \emptyset$ . Set  $B'_i := \gcd(\mathcal{B} \setminus \{d'_i\})$  for  $1 \leq i \leq 4$ , then  $B'_3 = B'_4 = 1$  and  $B'_i d'_i \notin \mathbb{N}(\mathcal{B} \setminus \{d'_i\})$  for  $i = 3, 4$ . Let  $d', h' \in \mathbb{Z}^+$  be such that  $d'_2 = h' d'_1 + d'$  and  $d'_3 = d'_2 + d'$ . If both  $d'_1$  and  $d'_4$  are even, then  $B'_2 = 2$  and  $2d'_2 = h' d'_1 + d'_3$ , thus  $\mathcal{B}_{red} = \emptyset$  if and only if  $\mathcal{C}_{red} = \emptyset$ , where  $\mathcal{C} = \{d_1, d_3, d_4\}$ . In case  $d'_1$  or  $d'_4$  is odd, we have that  $B'_2 = 1$  and  $B'_2 d'_2 \notin \mathbb{N}\{d'_1, d'_3, d'_4\}$ . Concerning  $B'_1$ , we have that

$B'_1 = \gcd\{h', d', d'_4\}$ ; since  $B'_1 d'_1 \mid h' d'_1 < d'_2 < d'_3$  then  $B'_1 d'_1 \in \mathbb{N}\{d'_2, d'_3, d'_4\}$  if and only if  $d'_4 \mid B'_1 d'_1$ . If  $d'_4 \nmid B'_1 d'_1$  then we can conclude that  $\mathcal{B}_{red} \neq \emptyset$  and  $I_{\mathcal{A}}$  is not a complete intersection. Otherwise,  $\mathcal{B}_{red} = \mathcal{B}'_{red}$  with  $\mathcal{B}' := \{d'_2, d'_3, d'_4\}$ , but  $\{d'_4, d'_2, d'_3\}$  is a generalized arithmetic sequence and  $d'_4$  is odd, then by Proposition 2.4 and Theorem 3.5,  $\mathcal{B}'_{red} \neq \emptyset$ .  $\square$

**Remark 3.10.** Furthermore, whenever  $I_{\mathcal{A}}$  is a complete intersection, if we let  $h$  be the integer such that  $\{hd_1, d_2, d_3\}$  is an arithmetic sequence, i.e.,  $h = (2d_2 - d_3)/d_1$ , we have the following results (see Lemma 2.1 and Proposition 2.2):

1. If  $d_1/\gcd\{d_1, d_2\}$  is even,  $\gcd\{d_1, d_2\}d_4 \in \mathbb{N}\{d_1, d_2, d_3\}$  and we take  $\beta_1, \beta_2, \beta_3 \in \mathbb{N}$  such that  $bd_4 = \beta_1 d_1 + \beta_2 d_2 + \beta_3 d_3$ , where  $b := \gcd\{d_1, d_2\}$ , then

- $I_{\mathcal{A}} = \left(x_4^b - x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}, x_2^2 - x_1^h x_3, x_1^{d_3/2b} - x_3^{d_1/2b}\right)$ ; and
- $g(\mathcal{S}) = (d_1 d_3 / 2b) - d_1 + d_2 - d_3 + (b - 1)d_4$ .

2. If both  $d_1$  and  $d_4$  are even,  $\mathcal{C}_{red} = \emptyset$  with  $\mathcal{C} := \{d_1, d_3, d_4\}$  and we denote by  $\mathcal{S}'$  the numerical semigroup  $\mathbb{N}(d_1/2) + \mathbb{N}(d_3/2) + \mathbb{N}(d_4/2)$ , then

- $I_{\mathcal{A}} = (x_2^2 - x_1^h x_3) + I_{\mathcal{C}} \cdot k[x_1, x_2, x_3, x_4]$ ; and
- $g(\mathcal{S}) = 2g(\mathcal{S}') + d_2$ .

## 4 Complete intersections in Fibonacci sequences.

Given  $p, q \in \mathbb{Z}^+$  with  $\gcd\{p, q\} = 1$ , our next aim is to characterize when  $I_{\mathcal{A}}$  is a complete intersection where  $\mathcal{A} = \{d_1, \dots, d_n\} \subset \mathbb{Z}^+$  with  $d_i = F_{e_i}$  for  $1 \leq i \leq n$ ,  $\{F_n\}$  denotes the  $(p, q)$ -Fibonacci sequence and  $\{e_1, \dots, e_n\}$  is a generalized arithmetic sequence. In order to achieve the characterization we first introduce some basic properties of linear second order recurrence sequences, focusing specially on those properties of the  $(p, q)$ -Fibonacci sequence and also in the Lucas one. Some of these properties are straight generalizations of those in [22], some others can be found in [14, Section 5] and the rest can be easily proved.

We denote by  $[a]_2$  the 2-valuation of  $a \in \mathbb{Z}^+$ , i.e.,  $[a]_2 := \max\{t \in \mathbb{N} : 2^t \mid a\}$ , and have the following result:

**Lemma 4.1.** [14, Theorems  $\bar{f}$ ,  $\bar{\ell}$  y  $\bar{f}\bar{\ell}$ ] Let  $a, b$  be positive integers and set  $d := \gcd\{a, b\}$ . Then,

- $\gcd\{F_a, F_b\} = F_d$ .
- $\gcd\{L_a, L_b\} = \begin{cases} L_d & \text{if } [a]_2 = [b]_2 \\ 2 & \text{if } [a]_2 \neq [b]_2, p, q \text{ are odd and } 3 \mid d \\ 2 & \text{if } [a]_2 \neq [b]_2, p \text{ is even and } q \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$

$$\bullet \gcd\{L_a, F_b\} = \begin{cases} L_d & \text{if } [a]_2 < [b]_2 \\ 2 & \text{if } [a]_2 \geq [b]_2, p, q \text{ are odd and } 3|d \\ 2 & \text{if } [a]_2 \geq [b]_2, p, b \text{ are even and } q \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

Observing Lemma 4.1 and that every  $(p, q)$ -Fibonacci sequence is strictly increasing except for  $p = 1$  (because  $F_1 = F_2$ ), and that every  $(p, q)$ -Lucas sequence is strictly increasing except for  $p \in \{1, 2\}$  (because  $L_0 \geq L_1$ ), we get the following divisibility properties.

**Corollary 4.2.** *Let  $a, b$  be two positive integers, we have the following properties:*

- (1) *If  $a \mid b$  then  $F_a \mid F_b$ .*
- (2) *If  $F_a \mid F_b$  then  $a \mid b$ , unless if  $p = b = 1$  and  $a = 2$ .*
- (3) *If  $a \geq 2$ , then  $L_a \mid L_b$  if and only if  $b/a$  is odd.*
- (4) *If  $b$  is even, then  $\gcd\{L_a, L_{a+b}\} = \gcd\{L_a, F_b\}$ .*

Denote by  $\{U_n\}_{n \in \mathbb{N}}$  any sequence satisfying that  $U_{n+2} = pU_{n+1} + qU_n$  for all  $n \geq 2$  and  $U_0, U_1 \in \mathbb{N}$  are not both null. In the following result we provide some properties of these sequences that we will use in the sequel, all of them are easy to prove. Property (4) can be found in [14, Propositions 5.1 and 5.2], properties (1) and (3) are straight generalizations of the corresponding results for  $p = q = 1$  one can find in [22] and (2) can be easily proved.

**Lemma 4.3.** *Let  $a, b, c, d, e$  be positive integers. Then,*

- (1)  $U_{a+b} = F_a U_{b+1} + q F_{a-1} U_b$ .
- (2) *If  $b \in \mathbb{N}\{a_1, \dots, a_k\}$  where  $a_1, \dots, a_k \in \mathbb{Z}^+$ , then  $F_b \in \mathbb{N}\{F_{a_1}, \dots, F_{a_k}\}$ .*
- (3)  $F_a U_b - F_c U_d = (-1)^e q^e (F_{a-e} U_{b-e} - F_{c-e} U_{d-e})$  if  $a + b = c + d$  and  $e \leq \min\{a, b, c, d\}$ .
- (4)  $L_a = F_{2a}/F_a = F_{a+1} + q F_{a-1}$ .
- (5)  $U_{a+2b} + (-1)^b q^b U_a = L_b U_{a+b}$ .

With these basic properties, the following inequalities are easy to prove.

**Corollary 4.4.** *Let  $a, b, c, d$  be positive integers. The following inequalities hold:*

- (1)  $q^b U_a \leq U_{a+2b}$  and equality holds if and only if  $a = U_1 = 0, b = 1$ .
- (2)  $L_a < F_{a+2}$ .
- (3)  $U_{a+b-2} < F_a U_b < U_{a+b-1}$  if  $a, b \geq 2$ .
- (4)  $F_a U_b < F_c U_d$  if  $a + b < c + d$ .
- (5) *If  $a < c, a < d$  and  $a + b = c + d$ , then  $F_a U_b < F_c U_d$  if and only if  $a$  is even.*

$$(6) \quad L_{a+b-1} < L_a L_b < \min\{L_{a+b+1}, 2L_{a+b}\}.$$

$$(7) \quad \text{If } a \leq b, \text{ then } L_a L_b < L_{a+b} \text{ if } a \text{ is odd and } L_a L_b > L_{a+b} \text{ if } a \text{ is even.}$$

Let  $d_i$  denote the  $e_i$ -th term of the  $(p, q)$ -Fibonacci sequence, where  $\{e_1, \dots, e_n\}$  is a generalized arithmetic sequence. This is, there exist  $h, a, d \in \mathbb{Z}^+$  such that  $d_1 := F_a$  and  $d_i := F_{ha+(i-1)d}$  for all  $i \geq 2$ . As we have mentioned, we aim at characterizing when  $I_{\mathcal{A}}$  is a complete intersection in terms of the values of  $p, q, n, h, a, d$ . This objective is achieved with the following result.

**Theorem 4.5.** *Let  $p, q \in \mathbb{Z}^+$  be two relatively prime integers and let  $\{F_n\}_{n \in \mathbb{N}}$  be the  $(p, q)$ -Fibonacci sequence. Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be the set with  $d_1 := F_a$  and  $d_i := F_{ha+(i-1)d}$  for all  $i \in \{2, \dots, n\}$  where  $h, a, d \in \mathbb{Z}^+$  and  $n \geq 3$ . Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff$  one the following holds:*

- (a)  $d$  is odd,
- (b)  $d \geq a$ ,
- (c)  $a = 2d$ ,
- (d)  $\gcd\{a, d\} = a - d$  and  $a$  is odd, or
- (e)  $n = 3$  and  $2d \mid a$ .

To prove this theorem we use two previous results, namely Lemma 4.6 and Proposition 4.8. The first one characterizes when  $d_3 \in \mathbb{N}\{d_1, d_2\}$  and also proves that whenever  $d_3 \in \mathbb{N}\{d_1, d_2\}$ , then  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 3$ , which particularly implies that  $I_{\mathcal{A}}$  is a complete intersection. The second one characterizes when  $I_{\mathcal{A}}$  is a complete intersection whenever  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ .

**Lemma 4.6.**  $d_3 \in \mathbb{N}\{d_1, d_2\} \iff d$  is odd or  $F_{2d} \geq \text{lcm}\{d_1, F_d\}$ . Moreover, if  $d_3 \in \mathbb{N}\{d_1, d_2\}$ , then  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 3$ .

**Proof.** If  $d$  is odd, by (5) in Lemma 4.3 we get that  $d_3 = (q^d F_{ha}/F_a)d_1 + L_d d_2 \in \mathbb{N}\{d_1, d_2\}$  and that  $d_{i+2} = L_d d_{i+1} + q^d d_i$  for all  $i \geq 2$ , thus  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 3$ . Suppose that  $d$  is even, set  $e := \gcd\{d_1, d_2\} = F_{\gcd\{a, d\}}$  and consider the numerical semigroup  $\mathcal{S} := \mathbb{N}\{d_1/e, d_2/e\}$ , its Frobenius number is  $g(\mathcal{S}) = ((d_1 d_2)/e - d_1 - d_2)/e$  (see, e.g. [19, Theorem 2.1.1]). If  $d \geq a$ , then  $eg(\mathcal{S}) < (d_1 d_2)/e \leq d_1 d_2 \leq F_d d_2 < d_i$  for all  $i \geq 3$  and we conclude that  $d_i \in \mathbb{N}\{d_1, d_2\}$ . Note that in this case  $F_{2d} > d_1 F_d \geq \text{lcm}\{d_1, F_d\}$ . Suppose now that  $a > d$  and let us study the existence of solutions  $(x, y) \in \mathbb{N}^2$  to the linear diophantine equation

$$d_1 x + d_2 y = d_3. \tag{1}$$

By Lemma 4.3 we know that  $\left(-q^d \frac{F_{ha}}{d_1}, L_d\right) \in \mathbb{Z}^2$  is an integer solution to the equation; thus the set of integral solutions is

$$\left\{ \left( -q^d \frac{F_{ha}}{d_1} + \lambda \frac{d_2}{e}, L_d - \lambda \frac{d_1}{e} \right), \lambda \in \mathbb{Z} \right\}.$$

We claim that  $-q^d \frac{F_{ha}}{d_1} + \frac{d_2}{e} > 0$ ; indeed since  $a > d$ , then  $e = F_{\gcd\{a,d\}} \leq F_{a-d}$  and  $q^d e F_{ha} = q^{d/2} e q^{d/2} F_{ha} < q^{d/2} F_{a-d} d_2 < d_1 d_2$ . Therefore (1) has a nonnegative integer solution if and only if  $d_1/e \leq L_d = F_{2d}/F_d$ , which is equivalent to  $F_{2d} \geq \text{lcm}\{d_1, F_d\}$ . Finally we have that

$$g(\mathcal{T})e \leq \frac{d_1 d_2}{e} \leq L_d d_2 < F_{d+2} d_2 < F_{ha+2d+1} < d_i \text{ for all } i \geq 4$$

and  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 4$ .  $\square$

Now we study when  $I_{\mathcal{A}}$  is a complete intersection provided  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ .

**Lemma 4.7.** *If  $2d \mid a$  and  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ , then  $d_4 \notin \mathbb{N}\{d_1, d_2, d_3\}$ .*

**Proof.** Assume that there exist  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$  such that  $d_4 = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3$ . Since  $d_4 = -q^d d_2 + L_d d_3$ , we have that  $\beta d_3 = \alpha_1 d_1 + (\alpha_2 + q^d) d_2$  where  $\beta := L_d - \alpha_3$  and the equation  $\beta d_3 = x d_1 + y d_2$  has a solution  $(x, y) \in \mathbb{N} \times \mathbb{N}$ . Moreover, the equality  $\beta d_3 = -\beta q^d (F_{ha}/d_1) d_1 + \beta L_d d_2$  yields that the set of integer solutions to this equation is

$$\left\{ \left( -\beta q^d \frac{F_{ha}}{d_1} + \lambda \frac{d_2}{F_d}, \beta L_d - \lambda \frac{d_1}{F_d} \right), \lambda \in \mathbb{Z} \right\}.$$

Nevertheless, for all  $\lambda > 0$  we have that

$$\beta L_d - \lambda \frac{d_1}{F_d} \leq (L_d)^2 - \frac{d_1}{F_d} \leq (L_d)^2 - \frac{F_{4d}}{F_d} = (L_d)^2 - L_{2d} L_d < 0;$$

and we can conclude that there is no solution  $(x, y) \in \mathbb{N} \times \mathbb{N}$ , a contradiction.  $\square$

**Proposition 4.8.** *If  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ , then  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 3$  and  $2d \mid a$ .*

**Proof.** By Lemma 4.6,  $d$  is even and by Lemma 4.3 we have that  $d_3 + q^d F_{ha} = L_d d_2$  and  $d_{i+1} + q^d d_{i-1} = L_d d_i$  for  $3 \leq i \leq n-1$ , which implies that  $m_i \leq L_d$  for all  $i \in \{2, \dots, n-1\}$ . We claim that  $m_2 d_2, \dots, m_n d_n$  are all different. Indeed if we assume that there exist  $i, j : 2 \leq i < j \leq n$  such that  $m_i d_i = m_j d_j$ , then  $m_i d_i \leq L_d d_i = d_{i+1} + q^d d_{i-1} < 2 d_{i+1}$ ; which implies that  $j = i+1$  and  $m_i d_i = d_{i+1}$  and  $d_i \mid d_{i+1}$ , but this is not possible because  $ha + (i-1)d \nmid ha + id$ . Hence by Proposition 2.6,  $I_{\mathcal{A}}$  is a complete intersection if and only if  $\mathcal{A}_{red} = \emptyset$ .

Let  $\mathcal{A}'$  be the minimal set of generators of  $\mathbb{N}\mathcal{A}$ , then there exists  $k \in \mathbb{N}$  and  $4 \leq i_1 < \dots < i_k \leq n$  such that  $\mathcal{A}' = \{d_1, d_2, d_3, d_{i_1}, \dots, d_{i_k}\}$ . We set

$$B'_j := \frac{\gcd(\mathcal{A}' \setminus \{d_j\})}{\gcd(\mathcal{A}')} \text{ for all } j \in \{1, 2, 3, i_1, \dots, i_k\}$$

and have that  $B'_j = 1$  for all  $j \in \{3, i_1, \dots, i_k\}$  because  $\gcd\{d_1, d_2\} = \gcd(\mathcal{A}')$  and  $B'_1 = \frac{F_{\gcd\{ha,d\}}}{\gcd(\mathcal{A}')}$ . Moreover  $B'_1 d_1 \notin \mathbb{N}(\mathcal{A}' \setminus \{d_1\})$  because  $B'_1 d_1 \leq F_d d_1 < d_i$  for all  $i \geq 2$ . If

$[a]_2 \leq [d]_2$  or there exists  $j \in \{1, \dots, k\}$  such that  $i_j$  is even, we also have that  $B'_2 = 1$ , this implies that  $\mathcal{A}_{red} \neq \emptyset$  and  $I_{\mathcal{A}}$  is not a complete intersection. If  $[a]_2 > [d]_2$  and  $i_j$  is odd for all  $j \in \{1, \dots, k\}$ , then  $\gcd\{a, 2d\} = 2\gcd\{a, d\}$  and  $B_2 = L_{\gcd\{a, d\}}$ . Suppose first that  $\gcd\{a, d\} < d$ , then  $B_2 d_2 \leq L_{d-1} d_2 < L_{ha+2d-2} < d_i$  for all  $i \in \{3, \dots, n\}$  and we claim that  $d_1 \nmid B_2 d_2$ ; otherwise we take  $\alpha_1, \alpha_2 \in \mathbb{Z}$  such that  $d_3 = \alpha_1 d_1 + \alpha_2 d_2$ , then  $B_2 \mid \alpha_2$  and we get that  $d_1 \mid d_3$ , a contradiction. Hence  $\mathcal{A}_{red} \neq \emptyset$  and  $I_{\mathcal{A}}$  is not a complete intersection. Finally assume that  $[a]_2 > [d]_2$  and that  $\gcd\{a, d\} = d$  or, equivalently, that  $2d \mid a$ ; if  $n \geq 4$ , by Lemma 4.7 we have that  $d_4 \notin \mathbb{N}\{d_1, d_2, d_3\}$ , then  $i_1 = 4$  and we are in the previous case; if  $n = 3$ , then  $L_d d_2 = q^d F_{ha} + d_3 \in \mathbb{N}d_1 + \mathbb{N}d_3$ ,  $\mathcal{A}_{red} = \emptyset$  and by Proposition 2.4 we conclude that  $I_{\mathcal{A}}$  is a complete intersection.  $\square$

*Proof of Theorem 4.5.* As we proved in Lemma 4.6,  $d_3 \in \mathbb{N}\{d_1, d_2\}$  if and only if  $d$  is odd or  $F_{2d} \geq \text{lcm}\{d_1, F_d\}$ . Moreover,  $F_{2d} \geq \text{lcm}\{d_1, F_d\} \iff F_{2d} F_{\gcd\{a, d\}} \geq d_1 F_d$ , and by (4) and (5) in Corollary 4.4 this is equivalent to  $\gcd\{a, d\} > a - d$  or  $a = 2d$  or  $\gcd\{a, d\} = a - d$  and  $a - d$  is odd. Furthermore,  $\gcd\{a, d\} > a - d$  if and only if  $d \geq a$ . So, we have that  $d_3 \in \mathbb{N}\{d_1, d_2\}$  if and only if  $d$  is odd,  $d \geq a$ ,  $a = 2d$  or  $\gcd\{a, d\} = a - d$  and  $a$  is odd. In this situation we also have that  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \in \{3, \dots, n\}$  and by Lemma 2.1 we conclude that  $I_{\mathcal{A}}$  is a complete intersection. If  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ , the result follows from Proposition 4.8.  $\square$

It is worth to mention that the characterization obtained in Theorem 4.5 does not depend on the values of  $p, q$  or  $h$ , but only on those of  $a, d$  y  $n$ .

**Remark 4.9.** Denoting  $e := F_{\gcd\{a, d\}}$  and  $\mathcal{S} := \sum_{i=1}^n \mathbb{N}(d_i/e)$ , whenever  $I_{\mathcal{A}}$  is a complete intersection, we get the following additional information (see Lemma 2.1 and Proposition 2.2):

1. If  $d$  is odd. Then,

- $I_{\mathcal{A}} = \left( x_1^{d_2/e} - x_2^{d_1/e}, x_3 - x_1^{q^d F_{ha}/d_1} x_2^{L_d}, x_4 - x_2^{q^d} x_3^{L_d}, \dots, x_n - x_{n-2}^{q^d} x_{n-1}^{L_d} \right); y$
- $g(\mathcal{S}) = \frac{1}{e} (d_1 d_2/e - d_1 - d_2).$

2. If  $d \geq a$ , or  $a = 2d$ , or  $\gcd\{a, d\} = a - d$  and  $a$  is odd. Then,

- $I_{\mathcal{A}} = \left( x_1^{d_2/e} - x_2^{d_1/e}, x_3 - x_1^{b_{3,1}} x_2^{b_{3,2}}, x_4 - x_1^{b_{4,1}} x_2^{b_{4,2}}, \dots, x_n - x_1^{b_{n,1}} x_2^{b_{n,2}} \right),$   
where  $b_{3,1}, \dots, b_{n,2} \in \mathbb{Z}^+$  satisfy that  $b_{i,1} d_1 + b_{i,2} d_2 = d_i$  for all  $i \in \{3, \dots, n\}$ ;  
and
- $g(\mathcal{S}) = \frac{1}{e} (d_1 d_2/e - d_1 - d_2).$

3. If  $n = 3$ ,  $2d \mid a$ ,  $a \neq 2d$  and  $d$  is even. Then  $e = F_d$  and

- $I_{\mathcal{A}} = \left( x_1^{d_3/F_{2d}} - x_3^{d_1/F_{2d}}, x_2^{L_d} - x_1^{q^d F_{ha}/d_1} x_3 \right); \text{ and}$
- $g(\mathcal{S}) = \frac{1}{F_d} (d_1 d_3/F_{2d} - d_1 + (L_d - 1) d_2 - d_3).$

## 5 Complete intersections in Lucas sequences.

Let  $p, q \in \mathbb{Z}^+$  be two relatively prime integers, in this section  $d_i$  denotes the  $e_i$ -th term of the  $(p, q)$ -Lucas sequence, where  $\{e_i\}_{1 \leq i \leq n}$  is an arithmetic sequence. This is, there exist  $a, d \in \mathbb{Z}^+$  such that  $d_i := L_{a+(i-1)d}$  for all  $i \geq 1$  and we aim at characterizing when  $I_{\mathcal{A}}$  is a complete intersection in terms of the values of  $p, q, n, a$  and  $d$ . This objective is achieved in Theorem 5.1. Recall that  $[a]_2$  (respect.  $[d]_2$ ) denotes the 2-valuation of  $a$  (respect.  $d$ ).

**Theorem 5.1.** *Let  $p, q$  be two relatively prime positive integers and let  $\{L_n\}_{n \in \mathbb{N}}$  be the  $(p, q)$ -Lucas sequence. Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be the set with  $d_i := L_{a+(i-1)d}$  for all  $i \in \{1, \dots, n\}$  where  $a, d \in \mathbb{Z}^+$  and  $n \geq 3$ . Then,  $I_{\mathcal{A}}$  is a complete intersection  $\iff$  one of the following holds:*

- (a)  $d$  is odd,
- (b)  $d \geq a$ ,
- (c)  $\gcd\{a, d\} = a - d$  and  $[d]_2 > [a]_2 \geq 1$ ,
- (d)  $n = 3$ ,  $p$  and  $a/d$  are odd and  $q$  is even, o
- (e)  $n = 3$ ,  $3 \nmid d$  and  $p, q$  and  $a/d$  are odd.

To prove this theorem we have followed a completely analogous scheme to the one we used for Theorem 4.5 but taking into account the  $(p, q)$ -Lucas sequence properties shown in Lemmas 4.1 and 4.3 and Corollaries 4.2 and 4.4. For a sake of brevity we are not including here the proof of Theorem 5.1, nevertheless we will state Lemma 5.2 and Proposition 5.4, that are, respectively, the Lucas versions of Lemma 4.6 and Proposition 4.8 which we have used to prove Theorem 5.1. Lemma 5.2 provides a characterization of when  $d_3 \in \mathbb{N}\{d_1, d_2\}$  and states that whenever  $d_3 \in \mathbb{N}\{d_1, d_2\}$ , then  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 3$ , which, in particular, implies that  $I_{\mathcal{A}}$  is a complete intersection. Proposition 5.4 characterizes when  $I_{\mathcal{A}}$  is a complete intersection whenever  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ .

**Lemma 5.2.**  $d_3 \in \mathbb{N}\{d_1, d_2\}$  if and only if  $d$  is odd or  $F_{2d} \geq \text{lcm}\{d_1, F_d\}$ . Moreover, whenever  $d_3 \in \mathbb{N}\{d_1, d_2\}$ , then  $d_i \in \mathbb{N}\{d_1, d_2\}$  for all  $i \geq 3$ .

Now we study when  $I_{\mathcal{A}}$  is not a complete intersection provided  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ .

**Lemma 5.3.** If  $a/d$  is odd,  $\gcd\{d_1, d_2\} = 1$  and  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ , then  $d_4 \notin \mathbb{N}\{d_1, d_2, d_3\}$ .

**Proposition 5.4.** If  $d_3 \notin \mathbb{N}\{d_1, d_2\}$ , then  $I_{\mathcal{A}}$  is a complete intersection  $\iff n = 3$ ,  $a/d$  is odd and  $\gcd\{d_1, d_2\} = 1$ .

**Remark 5.5.** Denoting  $e := \gcd(\mathcal{A})$  and  $\mathcal{S} := \sum_{i=1}^n \mathbb{N}(d_i/e)$ , whenever  $I_{\mathcal{A}}$  is a complete intersection, we get the following additional information (see Lemma 2.1 and Proposition 2.2):

1. If  $d$  is odd. Then,

- $I_{\mathcal{A}} = \left( x_1^{d_2/e} - x_2^{d_1/e}, x_3 - x_1^{q^d} x_2^{L_d}, \dots, x_n - x_{n-2}^{q^d} x_{n-1}^{L_d} \right)$ ; and
- $g(\mathcal{S}) = \frac{1}{e} (d_1 d_2 / e - d_1 - d_2)$ .

2. If  $d \geq a$ , or  $\gcd\{a, d\} = a - d$  and  $[d]_2 > [a]_2 \geq 1$ . Then,

- $I_{\mathcal{A}} = \left( x_1^{d_2/e} - x_2^{d_1/e}, x_3 - x_1^{b_{3,1}} x_2^{b_{3,2}}, x_4 - x_1^{b_{4,1}} x_2^{b_{4,2}}, \dots, x_n - x_1^{b_{n,1}} x_2^{b_{n,2}} \right)$ , with  $b_{i,1}, b_{i,2} \in \mathbb{N}$  such that  $b_{i,1} d_1 + b_{i,2} d_2 = d_i$  for all  $i \in \{3, \dots, n\}$ ; and
- $g(\mathcal{S}) = \frac{1}{e} (d_1 d_2 / e - d_1 - d_2)$ .

3. If  $n = 3$ ,  $d$  is even,  $a/d$  and  $p$  are odd and, either  $q$  is even, or  $3 \nmid d$ . Then  $e = 1$  and

- $I_{\mathcal{A}} = \left( x_1^{d_3/L_d} - x_3^{d_1/L_d}, x_2^{L_d} - x_1^{q^d} x_3 \right)$ ; and
- $g(\mathcal{S}) = d_1 d_3 / L_d - d_1 + (L_d - 1) d_2 - d_3$ .

Theorem 5.1 depends on the values of  $a, d, n$  and on the parity of  $p$  and  $q$ , in contrast to the corresponding result for the Fibonacci sequence where the values of  $p, q$  and even of  $h$  do not play any role in the result (see Theorem 4.5). This dependence of "all" initial values gives the insight that a more general result where the complete intersection property for  $I_{\mathcal{A}}$  is characterized, where  $d_i = L_{e_i}$  with  $\{e_1, \dots, e_n\}$  a generalized arithmetic sequence, the value of  $h$  such that  $\{h e_1, e_2, \dots, e_n\}$  is an arithmetic sequence is relevant. The following example shows the relevance of the value of  $h \in \mathbb{Z}^+$ , and gives a taste of the difficulty that might have the more general case in which  $e_1, \dots, e_n$  is a generalized arithmetic sequence.

**Example 1.** Let  $\mathcal{A}_h$  be the set  $\{L_5, L_{h5+d}, L_{h5+2d}\}$ , where  $\{L_n\}_{n \in \mathbb{N}}$  is the  $(1, 1)$ -Lucas sequence and  $h \in \mathbb{Z}^+$ . For  $h = 1$  and  $h = 3$ , the sets  $\mathcal{A}_1 = \{L_5 = 11, L_6 = 18, L_7 = 29\}$  and  $\mathcal{A}_3 = \{L_5 = 11, L_{16} = 2207, L_{17} = 3571\}$  determine complete intersection toric ideals. Nevertheless, for  $h = 2$  and  $h = 4$ , the sets  $\mathcal{A}_2 = \{L_5 = 11, L_{11} = 199, L_{12} = 322\}$  and  $\mathcal{A}_4 = \{L_5 = 11, L_{21} = 24476, L_{22} = 39603\}$  determine two non complete intersection toric ideals.

## 6 Complete intersections in certain projective monomial curves

In this section we denote  $\mathcal{A} = \{d_1, \dots, d_n\} \subset \mathbb{Z}^+$  and  $d := \max\{d_1, \dots, d_n\}$  and consider

$$\mathcal{A}^* := \{a_1, \dots, a_{n-1}, a_n, a_{n+1}\} \subset \mathbb{N}^2,$$

where  $a_i := (d_i, d - d_i)$  for every  $i \in \{1, \dots, n\}$  and  $a_{n+1} := (0, d)$ .

The objective of this section is to characterize when  $I_{\mathcal{A}^*}$  is a complete intersection in terms of the set  $\mathcal{A}$ , where  $\mathcal{A}$  belongs to any of the families studied in the previous sections. For this purpose we use Algorithm CI-projective-monomial-curve of Table 1.



We begin studying when  $I_{\mathcal{A}^\star}$  is a complete intersection, where either  $\mathcal{A}$  or  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence and we assume without loss of generality that  $\mathbf{gcd}(\mathcal{A}) = 1$ .

**Theorem 6.1.** *Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be a generalized arithmetic sequence with  $n \geq 3$ . Then,  $I_{\mathcal{A}^\star}$  is a complete intersection  $\iff n = 3$ ,  $\mathcal{A}$  is an arithmetic sequence and  $d_1$  is even.*

**Proof.** ( $\Rightarrow$ ) Set  $h \in \mathbb{Z}^+$  such that  $\{hd_1, d_2, \dots, d_n\}$  is an arithmetic sequence. For every  $i \in \{3, \dots, n-1\}$  we have that  $B_i = 1$  and  $B_1 = \gcd\{d_1, h\}$ , thus  $B_1 d_1 \leq hd_1 < d_2 < \dots < d_n$  and  $B_1 a_1 \notin \sum_{j=2}^{n+1} \mathbb{N} a_j$ . If  $n \geq 4$  or  $d_1$  is odd, it follows that  $B_2 = 1$  and  $\mathcal{A}_{red}^\star \neq \emptyset$ , thus  $I_{\mathcal{A}^\star}$  is not a complete intersection by Theorem 2.7. Suppose now that  $n = 3$  and  $d_1$  even, then  $B_2 = 2$ . If  $h \geq 2$ , then  $B_2 d_2 \neq \alpha_1 d_1 + \alpha_3 d_3$  with  $\alpha_1, \alpha_3 \in \mathbb{N}$  and  $\alpha_1 + \alpha_3 \leq 2$  and, again by Theorem 2.7, we have that  $I_{\mathcal{A}^\star}$  is not a complete intersection.

( $\Leftarrow$ ) We have that  $B_2 = 2$  and  $2a_2 = a_1 + a_3$ , then by Theorem 2.7,  $I_{\mathcal{A}^\star}$  is a complete intersection.  $\square$

**Remark 6.2.** *Whenever  $I_{\mathcal{A}^\star}$  is a complete intersection, i.e., when  $n = 3$ ,  $\mathcal{A}$  is an arithmetic sequence and  $d_1$  is even, we get the following minimal set of generators of the toric ideal (see Lemma 2.1 and Proposition 2.2):*

$$I_{\mathcal{A}^\star} = \left( x_2^2 - x_1 x_3, x_1^{d_3/2} - x_3^{d_1/2} x_4^{d_2-d_1} \right).$$

Concerning when  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence, we can assume without loss of generality that  $\{d_n, d_2, \dots, d_{n-1}\}$  is not an arithmetic sequence. Otherwise we take  $d'_1 := d_n$ ,  $d'_i = d_i$  for all  $2 \leq i \leq n-1$  and  $d'_n = d_1$  and have that  $\mathcal{A} \setminus \{d'_n\} = \{d'_1, \dots, d'_{n-1}\}$  is an arithmetic sequence.

**Theorem 6.3.** *Let  $\mathcal{A} = \{d_1, \dots, d_n\}$  be a set such that  $\mathcal{A} \setminus \{d_n\}$  is a generalized arithmetic sequence with  $n \geq 4$ . Then,  $I_{\mathcal{A}^\star}$  is a complete intersection  $\iff n = 4$ ,  $\{d_1, d_2, d_3\}$  is an arithmetic sequence and one of the following holds:*

1.  $d_1 / \gcd\{d_1, d_2\}$  is even and  $\gcd\{d_1, d_2\} d_4 = \beta_1 d_1 + \beta_2 d_2 + \beta_3 d_3$  with  $\beta_1 + \beta_2 + \beta_3 \leq \gcd\{d_1, d_2\}$ , or
2.  $d_1, d_4$  are even and  $\mathcal{C}_{red}^\star = \emptyset$  with  $\mathcal{C} = \{d_1, d_3, d_4\}$ .

**Proof.** Let  $h \in \mathbb{Z}^+$  be such that  $\{hd_1, d_2, \dots, d_{n-1}\}$  is an arithmetic sequence. We divide the proof in two parts, if  $B_n a_n \in \sum_{j \in \{1, \dots, n-1, n+1\}} \mathbb{N} a_j$  or equivalently if  $B_n d_n = \sum_{j=1}^{n-1} \beta_j d_j$  with  $\sum_{j=1}^{n-1} \beta_j \leq B_n$  (see Remark 2.8), then by Proposition 2.2 and Lemma 2.1 it follows that  $I_{\mathcal{A}^\star}$  is a complete intersection if and only if lo is  $I_{\mathcal{A}^\star \setminus \{a_n\}}$ . It is easy to check that  $I_{\mathcal{A}^\star \setminus \{a_n\}} = I_{\mathcal{A}_1^\star}$  with  $\mathcal{A}_1 = \{d_1/B_n, \dots, d_{n-1}/B_n\}$ . Moreover  $\gcd(\mathcal{A}_1) = 1$  and  $\mathcal{A}_1$  is a generalized arithmetic sequence, then by Theorem 6.1 we conclude that  $I_{\mathcal{A}_1^\star}$  is a complete intersection if and only if  $n = 4$ ,  $d_1/B_4$  is even and  $\{d_1, d_2, d_3\}$  is an arithmetic sequence; note also that  $B_4 = \gcd\{d_1, d_2\}$ .

Suppose now that  $B_n a_n \notin \sum_{j \in \{1, \dots, n-1, n+1\}} \mathbb{N} a_j$ . We have that  $B_i = 1$  for  $i \in \{3, \dots, n-1\}$  and then  $B_i a_i \notin \sum_{j \in \{1, \dots, n+1\}} \mathbb{N} a_j$ . Let us study the values of  $B_1$  and  $B_2$ . Set  $h, r \in \mathbb{Z}^+$  such that  $d_i = h d_1 + (i-1)r$  for all  $i \in \{2, \dots, n-1\}$ , we have that  $B_1 = \gcd\{h, r, d_n\}$ , and then  $B_1 d_1 < d_2 < \dots < d_{n-1}$ , so  $B_1 d_1 \in \mathbb{N}\{d_2, \dots, d_n\}$  if and only if  $d_n \mid B_1 d_1$ . However, if  $d_n \mid B_1 d_1$ , by Proposition 2.2 and Lemma 2.1 we have that  $I_{\mathcal{A}^\star}$  is a complete intersection if and only if  $I_{\mathcal{B}^\star}$  so is, where  $\mathcal{B} := \{d_2, \dots, d_n\}$ , but  $d_n \mid B_1 d_1 \mid h d_1$  and  $\mathcal{B}$  is a generalized arithmetic sequence; therefore, by Theorem 6.1 this implies that  $n = 4$  and  $\{d_4, d_2, d_3\}$  is an arithmetic sequence, what we had assumed not to happen. Concerning  $B_2$ , if  $n \geq 5$ ,  $d_1$  is odd or  $d_4$  is odd then  $B_2 = 1$  and  $B_2 a_2 \notin \sum_{j \in \{1, \dots, n+1\}} \mathbb{N} a_j$ . Otherwise, i.e., if  $n = 4$  and  $d_1$  and  $d_4$  are even, we have that  $B_2 = 2$  and set  $\mathcal{C} := \{d_1, d_3, d_4\}$ . Moreover,  $2a_2 \in \mathbb{N}\{a_1, a_3, a_4, a_5\}$  if and only if  $2d_2 = \alpha_1 d_1 + \alpha_3 d_3 + \alpha_4 d_4$  with  $\alpha_1 + \alpha_3 + \alpha_4 \leq 2$ , and this can only happen if:

- (a)  $2d_2 = d_1 + d_3$ ,
- (b)  $2d_2 = d_1 + d_4$ , or
- (c)  $2d_2 = d_3 + d_4$ .

If (a) holds we have that  $\{d_1, d_2, d_3\}$  is an arithmetic sequence and  $I_{\mathcal{A}^\star}$  is a complete intersection if and only if so is  $I_{\mathcal{C}^\star}$ . In (b) we have that  $2d_2 = h d_1 + d_3 = d_1 + d_4$  and we deduce that  $(h-1)d_1 + d_3 = d_4$ , thus one derives that  $\mathcal{C}_{red}^\star \neq \emptyset$  and  $I_{\mathcal{A}^\star}$  is not a complete intersection. Indeed, denoting  $B'_i := \min\{b \in \mathbb{Z}^+ \mid b a_i \in \sum_{j \in \{1, 3, 4\}} \mathbb{Z} a_j\}$  for all  $i \in \{1, 3, 4\}$ , from the previous equality it follows that  $B'_3 = B'_4 = 1$  and that  $B'_1 \leq h-1$ , therefore  $B'_1 d_1 \notin \mathbb{N}\{d_3, d_4\}$ , due to  $B'_1 d_1 < d_3 < d_4$ . In (c) we have that  $\{d_4, d_2, d_3\}$  is an arithmetic sequence, which we had assumed not to happen.

In the rest of cases we get that  $B_i a_i \notin \mathbb{N}(\mathcal{A}^\star \setminus \{a_i\})$  for all  $i \in \{1, \dots, n\}$ , and then  $\mathcal{A}_{red}^\star \neq \emptyset$  and  $I_{\mathcal{A}^\star}$  is not a complete intersection.  $\square$

**Remark 6.4.** *Moreover, whenever  $I_{\mathcal{A}^\star}$  is a complete intersection, we get the following minimal sets of generators of  $I_{\mathcal{A}^\star}$  depending on the cases (see Lemma 2.1 and Proposition 2.2):*

1. *Set  $b = \gcd\{d_1, d_2\}$ , if  $n = 4$ ,  $\{d_1, d_2, d_3\}$  is an arithmetic sequence,  $d_1/b$  is even and  $b d_4 = \beta_1 d_1 + \beta_2 d_2 + \beta_3 d_3$  with  $\beta_1 + \beta_2 + \beta_3 \leq b$ , then*

$$I_{\mathcal{A}^\star} = \left( x_4^b - x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} x_5^{b - \sum \beta_i}, x_2^2 - x_1 x_3, x_1^{d_3/2b} - x_3^{d_1/2b} x_5^{(d_2-d_1)/b} \right).$$

2. *If  $n = 4$ ,  $\{d_1, d_2, d_3\}$  is an arithmetic sequence,  $d_1, d_4$  are even and  $\mathcal{C}_{red}^\star = \emptyset$  with  $\mathcal{C} = \{d_1, d_3, d_4\}$ , then*

$$I_{\mathcal{A}^\star} = (x_2^2 - x_1 x_3) + I_{\mathcal{C}^\star} \cdot k[x_1, x_2, x_3, x_4, x_5].$$

We finish this section studying when  $I_{\mathcal{A}^\star}$  is a complete intersection in the following families, where  $\mathcal{A} = \{d_1, \dots, d_n\}$ ,  $n \geq 3$ ,  $p, q \in \mathbb{Z}^+$  are relatively prime and either

- $\mathcal{A}$  consists of terms the  $(p, q)$ -Fibonacci sequence whose indices are a generalized arithmetic sequence, i.e., there exist  $h, a, d \in \mathbb{Z}^+$  such that  $d_1 = F_a$ ,  $d_i = F_{ha+(i-1)d}$  for all  $i \geq 2$ , or
- $\mathcal{A}$  consists of terms of the  $(p, q)$ -Lucas sequence whose indices are an arithmetic sequence, i.e., there exist  $a, d \in \mathbb{Z}^+$  such that  $d_1 = L_a$ ,  $d_i = L_{ha+(i-1)d}$  for all  $i \geq 2$ .

In both cases we characterize when  $I_{\mathcal{A}^\star}$  is a complete intersection by means of the input data; which are the values of  $p, q, n, a$  and  $d$  (and  $h$  for the Fibonacci sequence).

**Theorem 6.5.** *Let  $p, q \in \mathbb{Z}^+$  be relatively prime and let  $\{F_n\}_{n \in \mathbb{N}}$  be the  $(p, q)$ -Fibonacci sequence. Set  $\mathcal{A} = \{d_1, \dots, d_n\}$  with  $d_1 := F_a$  y  $d_i := F_{ha+(i-1)d}$  for all  $i \in \{2, \dots, n\}$  where  $h, a, d \in \mathbb{Z}^+$  and  $n \geq 3$ . Then,  $I_{\mathcal{A}^\star}$  is a complete intersection  $\iff n = 3$ ,  $h = 1$ ,  $d$  is even and  $2d \mid a$ .*

**Proof.** We begin by observing that  $B_i = 1$  for all  $i \geq 3$  (see Lemma 4.1) and  $B_1 = F_{\gcd\{ha, d\}}/F_{\gcd\{a, d\}}$  and thus  $B_1 d_1 \leq F_d d_1 < d_2 < \dots < d_n$ , which implies that  $B_1 a_1 \notin \sum_{j=2}^{n+1} \mathbb{N} a_j$ . Let us study the possible values of  $B_2$ . If  $n \geq 4$  or  $[d]_2 \geq [a]_2$ , then  $B_2 = 1$ , and if  $n = 3$  and  $[d]_2 < [a]_2$  then  $B_2 = F_{\gcd\{a, 2d\}}/F_{\gcd\{a, d\}} = L_{\gcd\{a, d\}}$ . If  $d \nmid a$ , then  $B_2 d_2 \leq L_{d-1} d_2 < L_{ha+2d-2} < d_3$  and  $B_2 d_2 \in \mathbb{N} d_1 + \mathbb{N} d_3$  if and only if  $B_2 d_2 = \alpha d_1$ , but in this case  $\alpha > B_2$  and, by Remark 2.8,  $B_2 a_2 \notin \mathbb{N} a_1 + \mathbb{N} a_3 + \mathbb{N} a_4$ . Suppose now that  $d \mid a$ , if  $d$  is odd then  $L_d d_2 = -q^d (F_{ha}/d_1) d_1 + d_3 < d_3$  and again we have that  $B_2 d_2 \in \mathbb{N} d_1 + \mathbb{N} d_3$  if and only if  $B_2 d_2 = \alpha d_1$ , and thus  $B_2 a_2 \notin \mathbb{N} a_1 + \mathbb{N} a_3 + \mathbb{N} a_4$ . Finally if  $d$  is even we have the inequality  $B_2 d_2 = q^d (F_{ha}/d_1) d_1 + d_3 < 2d_3$ , hence whenever  $B_2 d_2 = \alpha_1 d_1 + \alpha_3 d_3$  with  $\alpha_1, \alpha_3 \in \mathbb{N}$ , then  $\alpha_3 < 2$ ; and if  $\alpha_3 = 0$ , then  $\alpha_1 > B_2$ . As a consequence, by Remark 2.8, it follows that  $B_2 a_2 \in \mathbb{N} a_1 + \mathbb{N} a_3 + \mathbb{N} a_4$  if and only if  $q^d (F_{ha}/d_1) + 1 \leq L_d$ . If  $h > 1$ , then  $F_{ha}/d_1 \geq F_{2a}/F_a = L_a \geq L_d$  and if  $h = 1$  it follows that  $L_d = F_{d+1} + qF_{d-1} > q^{d/2} F_1 + q q^{d/2-1} F_1 = q^d$ , and the result follows.  $\square$

**Remark 6.6.** *Whenever  $I_{\mathcal{A}^\star}$  is a complete intersection, i.e., when  $\mathcal{A} = \{d_1, d_2, d_3\}$  with  $d_1 = F_a$ ,  $d_2 = F_{a+d}$ ,  $d_3 = F_{a+2d}$ ,  $d$  is even and  $2d \mid a$ , we get the following minimal set of generators of  $I_{\mathcal{A}^\star}$  (see Lemma 2.1 and Proposition 2.2):*

$$I_{\mathcal{A}^\star} = \left( x_1^{d_3/F_{2d}} - x_3^{d_1/F_{2d}} x_4^{(d_3-d_1)/F_{2d}}, x_2^{L_d} - x_1^{q^d} x_3 x_4^{L_d-q^d-1} \right).$$

**Theorem 6.7.** *Let  $p, q \in \mathbb{Z}^+$  be relatively prime and let  $\{L_n\}_{n \in \mathbb{N}}$  be the  $(p, q)$ -Lucas sequence. Set  $\mathcal{A} = \{d_1, \dots, d_n\}$  with  $d_i := L_{a+(i-1)d}$  for all  $i \in \{1, \dots, n\}$  where  $a, d \in \mathbb{Z}^+$  and  $n \geq 3$ . Then,  $I_{\mathcal{A}^\star}$  is a complete intersection  $\iff n = 3$ ,  $d$  is even,  $p$  and  $a/d$  are odd and, either  $q$  is even or  $3 \nmid d$*

**Proof.** We begin by observing that  $B_i = 1$  for all  $i \geq 3$  (see Lemma 4.1). Let us study the possible values of  $B_2$ , if  $n \geq 4$  or  $[d]_2 \neq [a]_2$ , then  $B_2 = 1$ , and if  $n = 3$  and  $[d]_2 = [a]_2$  then  $B_2 = \gcd\{d_1, d_3\} / \gcd(\mathcal{A}) = L_{\gcd\{a, d\}} / \gcd(\mathcal{A})$  and  $\gcd(\mathcal{A}) \in \{1, 2\}$ . If  $\gcd(\mathcal{A}) = 2$  or  $\gcd\{a, d\} < d$ , then  $B_2 d_2 < d_3$  and thus  $B_2 a_2 \notin \mathbb{N}a_1 + \mathbb{N}a_3 + \mathbb{N}a_4$ . Otherwise, if  $\gcd(\mathcal{A}) = 1$  and  $\gcd\{a, d\} = d$ , then  $B_2 d_2 = L_d d_2 = (-1)^d q^d d_1 + d_3$ , if  $d$  is odd then  $L_d d_2 < d_3$  and again we have that  $B_2 a_2 \notin \mathbb{N}a_1 + \mathbb{N}a_3 + \mathbb{N}a_4$ ; otherwise  $B_2 d_2 = q^d d_1 + d_3$  and  $q^d + 1 \leq L_d$ . From here we deduce that  $\mathcal{A}_{red}^* = \emptyset$  if and only if  $n = 3$ ,  $[d]_2 = [a]_2$ ,  $\gcd\{a, d\} = d$ ,  $\gcd\{d_1, d_2, d_3\} = 1$  and  $d$  is even, by Lemma 4.1 the result follows.  $\square$

**Remark 6.8.** Whenever  $I_{\mathcal{A}^*}$  is a complete intersection, i.e., when  $\mathcal{A} = \{d_1, d_2, d_3\}$  with  $d_1 = L_a$ ,  $d_2 = L_{a+d}$ ,  $d_3 = L_{a+2d}$ ,  $d$  is even,  $p$  and  $a/d$  are odd and, either  $q$  is even or  $3 \nmid d$ , we get the following minimal set of generators of  $I_{\mathcal{A}^*}$  (see Lemma 2.1 and Proposition 2.2) :

$$I_{\mathcal{A}^*} = \left( x_1^{d_3/L_d} - x_3^{d_1/L_d} x_4^{(d_3-d_1)/L_d}, x_2^{L_d} - x_1^{q^d} x_3 x_4^{L_d-q^d-1} \right).$$

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